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Dynamical systems on compact extremally disconnected spaces

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Abstract

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We consider extremally disconnected compact spaces together with the semigroups of all self-embeddings and the groups of all autohomeomorphisms. The characterizations of minimality and ergodicity of these dynamical systems are given. We show that each universal point in (X, Emb) must be a P-point. We extend the concept of Rudin-Frolík ordering to all compact extremally disconnected spaces and prove that ccc spaces have the Frolík property.

Keywords: Extremally disconnected, semigroup of embeddings, dynamical systems, Rudin-Frolík ordering.

1. Introduction

For a topological space X, let $\mathrm{Hom}(X)$ denote the group of all autohomeomorphisms on X under the operation of composition $\circ$. It is normal in Topological Dynamics to consider an *action* of a group G on a space X. We only consider the case where G is discrete. We shall also be interested in having a discrete semigroup act on a compact space. For this we will consider the semigroup $(C(X, X), \circ)$, where $C(X, X)$ is the set of all continuous functions from X into X and $f \circ g(x) = f(g(x))$ for $f, g \in C(X, X)$ and $x \in X$. We will also be interested in the subsemigroup $\mathrm{Emb}(X)$, consisting of all embeddings of X into itself.

Definition 1. Let $(S, \cdot)$ be a semigroup. A dynamical system (X, S, π) is a system consisting of

(1) a nonempty compact space X (sometimes called a phase space), and
(2) an action of the semigroup S on X; that is, a homomorphism $\pi : (S, \cdot) \to (C(X, X), \circ)$.

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In all cases, where $(S, \cdot)$ is a monoid with identity e, we will suppose that $\pi(e)$ is the identity function on X.

Naturally, two examples are $(X, \mathrm{Hom}(X), \pi)$ and $(X, \mathrm{Emb}(X), \pi)$ where, in both cases, the homomorphism π is the identity function.

Fix a dynamical system (X, S, π). If π is clear we will simply refer to the system (X, S). Similarly we may write $s(x)$ or $s_\pi(x)$ for $\pi(s)(x)$. Note that we do not demand that π is an embedding, i.e. it is possible to have that $s\restriction X = t\restriction X$ and yet $s \neq t$. The *orbit* of $x \in X$ is $O(x) = \{s(x)\colon s \in S\}$. For a set $A \subset X$ and $s \in S$, we let $s[A] = \{s(a)\colon a \in A\}$. Since we are not dealing only with groups, s^{-1} does not refer to the inverse of s but rather $s^{-1}[A] = \{x \in X\colon s(x) \in A\}$. The set $A \subset X$ is an *invariant set*, if $A \neq \emptyset$ and if, for every $s \in S$, $s[A] \subset A$. Note that an invariant closed subset, A, can itself be regarded as a phase space of a dynamical subsystem, namely (A, S, π_A), where $\pi_A(s) = \pi(s)\restriction A$, for each $s \in S$.

Let us introduce some basic notions from topological dynamics.

Definition 2. The system (X, S) is called

(i) minimal if there is no closed invariant proper subset of X;

(ii) ergodic (in the topological sense) if there is no closed invariant proper subset of X which has nonempty interior.

It is clear that a minimal system is ergodic.

Lemma 3. *For a system (X, S), the following are equivalent.*

(1) *(X, S) is minimal.*

(2) *$O(x)$ is dense in X for each $x \in X$.*

(3) *For every nonempty open $U \subset X$, $\bigcup_{s\in S} s^{-1}[U] = X$.*

Lemma 4. *For a system (X, S), the following are equivalent.*

(1) *(X, S) is ergodic.*

(2) *For every nonempty open $U \subset X$, $\bigcup_{s\in S} s^{-1}[U]$* is dense in *$X$.*

(3) *For any nonempty open $U, V \subset X$, there is an $s \in S$ such that $U \cap s^{-1}[V] \neq \emptyset$.*

A Boolean algebra B is called *homogeneous* if, given $u, v \in B - \{0, 1\}$, there is an automorphism of B which takes u to v. A Boolean algebra B is called *weakly homogeneous* if, given nonzero $u, v \in B$, there is an automorphism, say h, of B such that $h(u) \wedge v \neq 0$. It is interesting to note the connection with ergodicity. Indeed, it is clear that an algebra B is weakly homogeneous if and only if the dynamical system $(X, \mathrm{Hom}(X))$ is ergodic, where X is the Stone space of B.

Next we deal with homomorphisms of dynamical systems.

Definition 5. A homomorphism from the dynamical system (X, S, π) to (Y, S, σ) is a continuous function $f\colon X \to Y$ which commutes with the actions, i.e.

$$(\forall x \in X)\,(\forall s \in S)\, f(s_\pi(x)) = s_\sigma(f(x)).$$

If there is an onto homomorphism, then we say that (Y, S) is a factor of (X, S).

Proposition 6. *If (Y, S) is a minimal system and $f:(X, S)\to(Y, S)$ is a homomorphism, then f is onto.*

Definition 7. A system (X, S) is called a universal minimal dynamical system if it is minimal and every minimal (S-) system is a factor of (X, S).

For every semigroup, there is a uniquely determined universal minimal system (see Ellis [9]) and, moreover, the phase space of the universal minimal system is extremally disconnected (abbreviated ED) (for a proof see [1], see also [6]). This is our motivation for studying extremally disconnected dynamical systems.

For any compact ED space we have two natural dynamical systems: (X, Hom) and (X, Emb), where $\mathrm{Hom}=\mathrm{Hom}(X)$ and $\mathrm{Emb}=\mathrm{Emb}(X)$. The two are naturally related. The latter provides a natural generalization of the concept of the Rudin–Frolík ordering on $\beta\omega$. We deal with conditions concerning minimality and ergodicity. We define what it means for a space to have *Frolík's property* and to have a *universal point*. We also raise many interesting open questions arising from the investigation.

2. Preliminaries

Recall that a space is an extremally disconnected (ED) space if the closure of every open set is again open. Since we deal only with Hausdorff spaces, a compact ED space will be *zero-dimensional* because it will have a base consisting of clopen sets. The class of compact zero-dimensional spaces is, of course, dual to the class of Boolean algebras. We let $\mathrm{CO}(X)$ denote the Boolean algebra of clopen subsets of X and if B is a Boolean algebra, $\mathrm{St}(B)$ will denote the Stone space of B. Let us recall the following summary of the basics of Stone duality.

Proposition 8. *Let B be an infinite Boolean algebra and let $X=\mathrm{St}(B)$.*

(1) *$B\cong\mathrm{CO}(X)$ (hence we will consider them as identical).*

(2) *The weight of X, $w(X)$, is equal to the cardinality of B, where $w(X)$ is defined as the minimum cardinality of a base for the topology for X.*

(3) *For each $U\in\mathrm{CO}(X)-\{\emptyset\}$, the weight of the subspace U (if infinite) is equal to the cardinality of the factor algebra $B\restriction U=\{b\in B: b\leq U\}$.*

(4) *If Y is a compact subset of X, then $\varphi(U)=U\cap Y$ defines a homomorphism of B onto $\mathrm{CO}(Y)$.*

(5) *If φ is a homomorphism from B onto some Boolean algebra C, then there is a compact $Y\subset X$ such that $\mathrm{CO}(Y)\cong C$ (via the canonical isomorphism).*

(6) *If $C\subset B$ is a subalgebra, then $\mathrm{St}(C)$ is a continuous image of X.*

Proposition 9. *Suppose that X is a compact space and let $B=\mathrm{CO}(X)$. The following are equivalent.*

(1) *X is ED.*
(2) *B is a complete Boolean algebra (cBa).*
(3) $\mathrm{CO}(X) = \mathrm{RO}(X)$.
(4) *If $U \subset X$ is open, then $\bar{U} \in \mathrm{CO}(X)$.*
(5) *If U, V are disjoint open sets then $\bar{U} \cap \bar{V} = \emptyset$.*
(6) *Every open subset of X is extremally disconnected.*
(7) *If $Y \subset X$ is dense and f is a continuous function from Y into a compact space K, then there is a continuous (Čech–Stone) extension of f to all of X.*

For any space X, the algebra of regular open sets, $\mathrm{RO}(X)$, is a complete Boolean algebra (when ordered by inclusion), where

$$\mathrm{RO}(X) = \{U \subset X \colon U \text{ is open, and } U = \operatorname{int} \bar{U}\}.$$

The Stone space of $\mathrm{RO}(X)$ is extremally disconnected and it is usually denoted by $E(X)$ which is called the Gleason spaces of X. Let κ be an infinite cardinal. The power set of κ, $\mathcal{P}(\kappa)$, is a very important complete Boolean algebra for us. Also, let C_κ denote the free (Boolean) algebra with κ generators. The generalized Cantor discontinuum, 2^κ, is the Stone space of C_κ and $E(2^\kappa)$ is the Stone space of the completion of C_κ.

It is well known that an infinite Boolean algebra, B, is homogeneous if and only if $B \restriction u \cong B$ for each nonzero $u \in B$ [14]. We shall say that a Boolean algebra is homogeneous in cardinality (or any other cardinal function) if $|B \restriction u| = |B|$ for each nonzero $u \in B$. A topological space X will be said to be homogeneous in weight if $w(U) = w(X)$ for each nonempty open subset $U \subset X$. Therefore, a zero-dimensional space X is homogeneous in weight iff $\mathrm{CO}(X)$ is homogeneous in cardinality. The *character* of a point x in X is the minimum cardinality of a neighbourhood base for the point; by Stone duality, this notion corresponds to the minimum cardinality of a base for an ultrafilter on a Boolean algebra.

If Y is an arbitrary Tychonoff space, then βY denotes the Čech–Stone compactification of Y. Note that if Y is, in addition, an extremally disconnected space, then βY is simply the Stone space of $\mathrm{CO}(Y)$, and βY is ED. For each cardinal κ, $\beta\kappa$ denotes the Čech–Stone compactification of the discrete space κ and therefore, $\beta\kappa$ is the Stone space consisting of all ultrafilters on κ.

The study of the space $\beta\omega$, and its subspace $\omega^* = \beta\omega - \omega$, is so important to the study of compact ED spaces because of the following well-known fact.

Proposition 10. *Let X be a compact ED space.*

(1) *If $\{a_n \colon n \in \omega\}$ is a discrete subset of X, then the closure, $\overline{\{a_n \colon n \in \omega\}}$, is homeomorphic to $\beta\omega$.*

(2) *If A, B are σ-compact subsets of X, then $\bar{A} \cap \bar{B} \neq \emptyset$ implies that either $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$.*

If $p \in \beta\omega$ and $\{d_n: n \in \omega\} \subset X$, then p-$\lim\{d_n: n \in \omega\}$ is defined to be a point x in X such that $x \in \overline{\{d_n: n \in A\}}$ for each $A \in p$. We may say that x is a p-limit of $\{d_n: n \in \omega\}$. In case X is compact and Hausdorff, p-limits always exist and are unique.

We consider two countable discrete subsets of X and we record the following basic fact about ED spaces.

Proposition 11 (Frolík). *Suppose p, q are ultrafilters on ω. Let X be an ED space and suppose that $x \in X$ is a p-limit of a countable discrete set $\{a_n: n \in \omega\}$ and also a q-limit of a countable discrete set $\{b_n: n \in \omega\}$. Then either there is an $A \in p$ such that $\{a_n: n \in A\} \subset \overline{\{b_n: n \in \omega\}}$ or there is a $B \in q$ such that $\{b_n: n \in B\} \subset \overline{\{a_n: n \in \omega\}}$.*

Proof. Let

$$A = \{n \in \omega: a_n \in \overline{\{b_n: n \in \omega\}}\}$$

and

$$B = \{n \in \omega: b_n \in \overline{\{a_n: n \in \omega\}}\}.$$

Now, since X is an ED space, the sets $\{a_n: n \in \omega - A\}$ and $\{b_n: n \in \omega - B\}$ must have disjoint closures. Hence x is not in the closure of one of these sets. Therefore, either $A \in p$ or $B \in q$. □

We can remark that the above facts concerning countable discrete sets and σ-compact sets hold in a much wider class of spaces, namely compact F-spaces. A compact space is an F-space if disjoint open σ-compact subsets have disjoint closures. For example, ω^* is an F-space but it is not an ED space.

3. Minimal and ergodic

In this section we investigate the properties of minimality and ergodicity on the systems (X, Hom) and (X, Emb) when X is a compact ED space. Generally $1 \leq |\mathrm{Hom}(X)| \leq |X|$, and it can happen that $|\mathrm{Hom}(X)| = 1$ since there are rigid compact ED spaces. However, (X, Emb) is a much richer structure since $|\mathrm{Emb}(X)| = |X|$ for every compact ED space X—as we shall show. We begin with $\mathrm{Hom}(X)$. The first result does not need a proof.

Proposition 12. *If X is finite then (X, Hom) is minimal and $(X, \mathrm{Hom}) = (X, \mathrm{Emb})$.*

Theorem 13. *Let X be an infinite compact ED space.*

(1) *(X, Hom) is minimal*
- *iff the Boolean algebra $\mathrm{CO}(X)$ is homogeneous*
- *iff every nonempty clopen subset of X is homeomorphic to X.*

(2) *(X, Hom) is ergodic*
- *iff there is a homogeneous cBa A, such that $\mathrm{CO}(X) \cong \prod_{i \in I} A$ for some index set I*

- *iff there is a compact ED space Y such that $(Y, \mathrm{Hom}(Y))$ is minimal and $X \cong \beta[\sum_{i\in I} Y_i]$ where, for each $i \in I$, $Y_i \cong Y$ and $\sum Y_i$ denotes the topological sum operation.*

We shall delay presentation of the proof for the sake of continuity.

Corollary 14. *If X has an isolated point, then (X, Hom) is ergodic iff $X \cong \beta\kappa$ for some cardinal κ.*

Now we state the analogous result for $\mathrm{Emb}(X)$.

Theorem 15. *Let X be an infinite compact ED space.*

(1) *(X, Emb) is minimal*
- *iff $\mathrm{CO}(X)$ is homogeneous in cardinality*
- *iff X is homogeneous in weight.*

(2) *(X, Emb) is ergodic*
- *iff there is a cardinal τ and a family $\{B_i\colon i \in I\}$ of cBa's such that $\mathrm{CO}(X) \cong \prod_{i\in I} B_i$ and each B_i is homogeneous in cardinality τ*
- *iff there is a cardinal τ and a family of compact ED spaces $\{Y_i\colon i \in I\}$ such that $X \cong \beta[\sum_{i\in I} Y_i]$ and each Y_i is homogeneous in weight τ.*

One implication in each of the above two theorems is easy while the other implications are closely related to the following two results about complete Boolean algebras. The first is a fundamental and deep result of Solovay and Koppelberg [14] (independently). A proof of both results can be found in the Handbook of Boolean Algebras, the first is in [21] and the second is in [14].

Theorem 16 (Solovay-Koppelberg). *A complete Boolean algebra B is weakly homogeneous iff there is a homogeneous cBa A and a set I such that $B \cong \prod_{i\in I} A$.*

Theorem 17 [2]. *If B is an infinite cBa of cardinality κ, then the free algebra, with κ generators, $\mathscr{C}_\kappa$, is a subalgebra of B.*

It will be more convenient for us to restate Theorem 17 as follows.

Corollary 18 [2]. (1) *If B is a cBa of cardinality κ, then every cBa A of cardinality at most κ is a homomorphic image of B.*

(2) *If X is a compact ED space of weight κ, then every compact ED space Y of weight at most κ can be embedded into X.*

Proof. Clearly the second statement is just the dual of the first. In the infinite case, to deduce the first from 17, assume that A is a cBa with $|A| \leq \kappa$. Let $\mathscr{C}_\kappa \subset B$ be a copy of the free algebra. Since $\mathscr{C}_\kappa$ is free, we may choose a homomorphism, say φ, from $\mathscr{C}_\kappa$ onto A. Now, since A is complete, we may apply Sikorski's extension theorem to complete the proof. $\square$

Corollary 19 [2]. (1) *If B is an infinite complete Boolean algebra, then there are $2^{|B|}$ distinct homomorphisms from B onto itself.*

(2) *If X is an infinite compact ED space, then $|X| = |\mathrm{Emb}| = 2^{w(X)}$.*

Proof. Let $\kappa = |B|$ and choose generators, $\{b_\alpha : \alpha < \kappa\} \subset B$, for the free Boolean algebra. Every function from κ onto B gives rise to a homomorphism from the free algebra onto $|B|$. By Sikorski's extension theorem, all these 2^κ distinct homomorphisms extend to distinct homomorpisms on B. Since distinct homomorphisms from B onto itself give rise to distinct embeddings of $\mathrm{St}(B)$ into itself, we have $|\mathrm{Emb}| = 2^{w(X)}$. We have only left to prove that $|X| = 2^{w(X)} = 2^{|B|}$. Since there are certainly no more than $2^{|B|}$ ultrafilters on B, we have that $|X| \leqslant 2^{|B|}$. Finally, there are at least 2^κ ultrafilters on B, since for each subset A of $\{b_\alpha : \alpha < \kappa\}$ there is an ultrafilter $\mathcal{U}_A$ on B such that $\mathcal{U}_A \cap \{b_\alpha : \alpha < \kappa\} = A$. □

Corollary 20 [2]. *If X is an infinite compact ED space, then there is a point $x \in X$, such that the character of x is equal to $w(X)$.*

Proof of Theorem 13. We shall first establish the equivalences, in both cases, of the first two statements. The equivalence of the third statement with the second is then easily established using Stone duality.

We shall begin our proof with part (2). By our remark following Lemma 4, we see that $\mathrm{CO}(X) = B$ is weakly homogeneous if and only if (X, Hom) is ergodic. Part (2) then follows directly from Theorem 16. For part (1), assume that $\mathrm{CO}(X)$ is homogeneous and let us show that (X, Hom) is minimal. Let $x \in X$ and $U \in \mathrm{CO}(X) - \{\emptyset\}$ be arbitrary. By Lemma 3, we must show that there is an $h \in \mathrm{Hom}$ such that $h(x) \in U$. But this is trivial since if $x \notin U$ we may let $V = X - U$. From the definition of homogeneous, we know that there is a homeomorphism which maps V onto U. Finally, let us suppose that (X, Hom) is minimal. By part (2), there is a homogeneous cBa A, and a set I, such that $\mathrm{CO}(X) \cong \prod_{i \in I} A$. We finish by showing that $\mathrm{CO}(X) \cong A$. Choose $U \in \mathrm{CO}(X)$ such that $U \cong A$. By Lemma 3(3), we have that $X = \bigcup_{h \in \mathrm{Hom}} h^{-1}[U]$. By compactness, X is covered by finitely many homeomorphic copies of U. Since X is infinite and U is homogeneous, it follows that X is homeomorphic to U—hence $\mathrm{CO}(X) \cong A$. □

Proof of Theorem 15. (1) Suppose that (X, Emb) is minimal and let $\tau = w(X)$. By Corollary 20, there is a point $x \in X$, with character τ. Let U be a nonempty open subset of X. By the minimality of (X, Emb), there is an embedding f such that $f(x) \in U$. Since the character of x is τ, it follows that $w(f[X] \cap U) \geqslant \tau$. Also, $w(U) \leqslant w(X)$, hence we have that X is homogeneous in weight.

Now suppose that X is homogeneous in weight and let $U \in \mathrm{CO}(X) - \{\emptyset\}$ be arbitrary. Since $w(U) = w(X)$, there is, by Corollary 18, an embedding of X into U. Therefore the orbit of every point is dense and (X, Emb) is minimal.

(2) Suppose (X, Emb) is ergodic and let $\tau = \min\{w(U)\colon U \in \mathrm{CO}(X) - \{\emptyset\}\}$. Let $\{Y_i\colon i \in I\} \subset \mathrm{CO}(X)$, be a maximal disjoint family of clopen subsets of X with weight τ. A compact ED space is the Čech-Stone compactification of each of its dense subsets hence it is sufficient to show that $\bigcup_{i \in I} Y_i$ is dense. Let $V \in \mathrm{CO}(X) - \{\emptyset\}$. By Lemma 4(3), there is an $f \in \mathrm{Emb}$, such that $V \cap f^{-1}[Y_0] \neq \emptyset$. Since $V \cap f^{-1}[Y_0] \in \mathrm{CO}(X) - \{\emptyset\}$ has weight τ and the family $\{Y_i\colon i \in I\}$ is maximal, there is an $i \in I$ such that $Y_i \cap V \neq \emptyset$.

For the converse direction, suppose that $X \cong \beta[\bigcup_{i \in I} Y_i]$ so that each Y_i is homogeneous in weight τ. For each $i, j \in I$, and nonempty clopen $U_i \subset Y_i$ and $U_j \subset Y_j$, there is an $f \in \mathrm{Emb}$, such that $f[U_i] \subset U_j$. From this, it clearly follows that if U, V are nonempty open subsets of X, there is an $f \in \mathrm{Emb}$, such that $f^{-1}[V] \cap U \neq \emptyset$. Therefore (X, Emb) is ergodic. □

Let us illustrate the above concepts by considering the spaces $\beta\kappa - \kappa$. Although these spaces are not ED, they are certainly subspaces of ED spaces and are therefore compact zero-dimensional F-spaces.

Proposition 21. *The system* $(\beta\kappa - \kappa, \mathrm{Hom})$ *is minimal iff* $\kappa = \omega$, *or* ($\kappa = \omega_1$ *and* $\beta\omega - \omega \cong \beta\omega_1 - \omega_1$) *and there is no other possibility for* κ.

Let us note that the question of whether the two spaces $\beta\omega - \omega$ and $\beta\omega_1 - \omega_1$ can be homeomorphic was first posed by Turzanski. It was quickly popularized by Szymanski, Frankiewicz and Comfort. Until now, nobody has shown, in ZFC, that they are not. However it has been shown [3] that $\beta\omega_2 - \omega_2$ is not homeomorphic to $\beta\omega - \omega$. The observation follows easily from this last result.

Proposition 22. $(\beta\kappa - \kappa, \mathrm{Emb})$ *is minimal iff* $2^\omega = 2^\kappa$.

Proof. Suppose that $(\beta\kappa - \kappa, \mathrm{Emb})$ is minimal. It is a classical result of Pospíšil [18] that there is an ultrafilter p on κ with character 2^κ. Since the system is minimal, we may choose an embedding taking p into the clopen set ω^*. It then follows that the weight of ω^* is 2^κ. For the converse direction, note that if $2^\kappa = 2^\omega$, then, by Corollary 18, $\beta\omega$ contains a copy of $\beta\kappa$. □

4. Universal points

Let us begin this section with a very natural definition for an arbitrary dynamical system.

Definition 23. For a system (X, S, π), we call a point $x \in X$ a universal point, if the orbit of x is the entire space X, i.e. $O(x) = X$.

For an infinite compact ED space X, the system (X, Hom) has no universal point. This, of course, is only a different form of Frolík's theorem that an infinite compact

ED space is not homogeneous. However let us recall that Frolík's results give much more. Indeed, in the first place, he shows that if f is an embedding of X into itself, then the set of fixed points of f is clopen, see Theorem 29. Since we now have Corollary 18 at our disposal, we may choose an embedding f of X into itself so that $f[X]$ is nowhere dense in X. Choose $p \in X$ and let $q = f(p)$. Then there is no embedding of X into itself which takes q to p, for if g were such an embedding, the set of fixed points of the embedding $f \circ g$ would be a nonempty subset of the nowhere dense set $f[X]$ and therefore not clopen. So, the techniques developed for the system (X, Hom) also give us interesting information about (X, Emb). By a more detailed analysis of ultrafilters on ω Frolík was able to obtain such points without benefit of Balcar and Franek's theorem. Even more can be deduced from Kunen's work on inhomogeneity of F-spaces.

We will use some basic facts concerning the Rudin–Keisler, or RK, pre-ordering on ultrafilters on ω where $p \leqslant_{\mathrm{RK}} q$ means that for some function $f: \omega \to \omega$, $q = \{f^{-1}[A]: A \in p\}$. Moreover, we write $p \approx q$ if the function f can be chosen to be a permutation on ω. We write $p \not\approx q$ if there is no such permutation and say that p and q are RK distinct. Rudin [19], showed that if $p \leqslant_{\mathrm{RK}} q$ and $q \leqslant_{\mathrm{RK}} p$ then $p \approx q$. It is well known that if an ultrafilter $p \in \omega^*$ is RK minimal in ω^*, then p is a P-point in ω^* (see [5]).

A point p in a space X is a *weak P-point* if p is not the limit point of any countable subset of X. Kunen proved in [15], that there are 2^c RK distinct weak P-points in ω^*. See [16], for applications of weak P-points to nonhomogeneity of F-spaces.

Let us observe the following corollary to Proposition 11.

Corollary 24. *Suppose that p, q are free ultrafilters on ω and suppose that p is a weak P-point of ω^*. If a point x in an ED space X is both a p-limit and a q-limit of discrete subsets of X, then q is a p-limit of some discrete subset of $\beta\omega$. If, furthermore, q is a weak P-point of ω^*, then $p \approx q$.*

Proof. Fix discrete subsets $\{a_n: n \in \omega\}$ and $\{b_n: n \in \omega\}$ such that

$$x = p\text{-lim}\{a_n: n \in \omega\} = q\text{-lim}\{b_n: n \in \omega\}.$$

If there is an $A \in p$ so that $\{a_n: n \in A\} \subset \overline{\{b_n: n \in \omega\}}$, then x is a p-limit of some countable discrete subset of $\overline{\{b_n: n \in \omega\}}$. To see that in this case we are done, consider the function $\varphi(b_n) = n$ for $n \in \omega$. By our earlier remarks, φ extends to a homeomorphism $\varphi: \overline{\{b_n: n \in \omega\}} \to \beta\omega$. Since φ is continuous, we have that

$$\varphi(x) = q\text{-lim}\{b_n: n \in \omega\} = q\text{-lim}\{n: n \in \omega\} = q.$$

Therefore $q = p\text{-lim}\{\varphi(a_n): n \in A\}$.

If there is no A as above, then by Proposition 11, we deduce that there is a $B \in q$ such that $\{b_n: n \in B\} \subset \overline{\{a_n: n \in \omega\}}$. Let

$$B_0 = \{n \in B: b_n \in \{a_n: n \in \omega\}\}.$$

If $B_0 \in q$, then the function on B_0 defined by $\varphi(n) = k$ where $b_n = a_k$ is easily modified to give a permutation on ω which takes q to p.

Let $B_1 = B - B_0$, we are finished by showing that $B_1 \notin q$. But now $\{b_n: n \in B_1\}$ is a countable subset of $\overline{\{a_n: n \in \omega\}} - \{a_n: n \in \omega\}$ and, similar to the above, x cannot be the limit point of any countable subset of $\overline{\{a_n: n \in \omega\}} - \{a_n: n \in \omega\}$ since p is a weak P-point.

Now, if q is also a weak P-point, then p is a q-limit of a countable discrete subset of $\beta\omega$. But, of course, this discrete subset must essentially be a subset of ω, we obtain the desired permutation as above. □

The immediate interest to us is the following application to the system (X, Emb).

Corollary 25. *If X is an infinite compact ED space then, in the system (X, Emb), there are $p,q \in X$ so that $O(p) \cap O(q) = \emptyset$. In fact, there are at least $2^{\mathfrak{c}}$ many pairwise disjoint orbits.*

Proof. Since X is an infinite compact Hausdorff space, there is an infinite discrete subset $\{a_n: n \in \omega\}$ of X. Let $\{p_\alpha: \alpha < 2^{\mathfrak{c}}\}$ be a set of weak P-points of $\beta\omega - \omega$ so that $p_\alpha \not\approx p_\beta$ for $\alpha < \beta < 2^{\mathfrak{c}}$. If, for each $\alpha < 2^{\mathfrak{c}}$, we let $x_\alpha = p_\alpha\text{-}\lim\{a_n: n \in \omega\}$, then, for $\alpha < \beta < 2^{\mathfrak{c}}$, $O(x_\alpha) \cap O(x_\beta) = \emptyset$. This is true because embeddings preserve discreteness and so any point of $O(x_\alpha) \cap O(x_\beta)$ would be both a p_α-limit and a p_β-limit of discrete subsets of X, contradicting that $p_\alpha \not\approx p_\beta$. □

The above suggests that there is unlikely to be a universal point for (X, Emb) when X is a compact ED space. However we show that an isolated point is a universal point. We will also show that the question of the existence of a nonisolated universal point involves measurable cardinals. Let us remark that it does not hold for arbitrary compact spaces that an isolated point is universal. For example, a space consisting of a single converging sequence has no universal point.

Theorem 26. *Consider the system (X, Emb) for a compact ED space X.*

(1) *Each isolated point of X is a universal point.*

(2) *Each universal point of X is a P-point.*

Proof. (1) We can assume that X is infinite. Let $u \in X$ be isolated and suppose that $x \in X$. Let $\kappa = |\mathrm{CO}(X)|$ and choose, by Theorem 17, $\{V_\alpha: \alpha < \kappa\} \subset \mathrm{CO}(X)$ so that they are generators of the free algebra as a subalgebra of $\mathrm{CO}(X)$. Therefore the weight of both V_0 and its complement are κ. From this we may conclude that there is a $U \in \mathrm{CO}(X)$ so that $x \notin U$ and the weight of U is κ. By Corollary 18, there is an embedding, say g of X into U. Let f be the mapping of X into itself defined by $f(u) = x$ and $f(y) = g(y)$ for $y \in X - \{u\}$. Then $f \in \mathrm{Emb}$ and so $x \in O(u)$.

(2) Suppose that $u \in X$ is universal and is not a P-point. Then X must be infinite and have no isolated points. Choose an arbitrary discrete set $\{d_n: n \in \omega\} \subset X$.

(i) We first prove that u is not the limit of any countable discrete subset of X. Fix a pair, $p,q\in\omega^*$, of weak P-points of ω^* such that $p\not\approx q$. Let $x=p\text{-lim}\{d_n\colon n\in\omega\}$ and let $y=q\text{-lim}\{d_n\colon n\in\omega\}$. Suppose that $r\in\omega^*$ and that u is the r-limit of some countable discrete subset of X.

Now if $f\in\text{Emb}$ is such that $f(u)=x$, then x is both a p-limit and an r-limit of discrete sets. By Corollary 24, r is a p-limit of some countable discrete subset of $\beta\omega$. Similarly, r is a q-limit of some countable discrete subset of $\beta\omega$. But then again applying Corollary 24 to r in $\beta\omega$, we obtain that $p\approx q$, a contradiction.

(ii) Now we show that there must be a P-point in ω^*, in fact even more, there is a smallest element in ω^* with respect to the Rudin-Keisler pre-ordering.

Since u is not a P-point, we may fix a disjoint family $\{A_n\colon n\in\omega\}\subset\text{CO}(X)$ so that $u\in\overline{(\bigcup_{n\in\omega}A_n)}-(\bigcup_{n\in\omega}A_n)$. Let r be the unique ultrafilter on ω such that

$$r=\{Y\subset\omega\colon u\in\overline{\bigcup\{A_n\colon n\in Y\}}\}.$$

Claim. *The ultrafilter r is the RK minimum of ω^*. Therefore it must be a P-point.*

Proof. Let p be an arbitrary element of ω^* and let $x=p\text{-lim}\{d_n\colon n\in\omega\}$. Fix $f\in\text{Emb}$ so that $f(u)=x$. By part (i), the point $x=f(u)$ is not the limit point of any countable discrete subset of $f[X]$. Hence we can assume that $\{d_n\colon n\in\omega\}\cap f[X]=\emptyset$. Now we choose, by induction on $n\in\omega$, a disjoint family $\{U_n\colon n\in\omega\}\subset\text{CO}(X)$ such that $U_n\cap f[X]=f[A_n]$ and $d_n\in\bigcup_{k\leq n}U_k$. Now define a function $h\colon\omega\to\omega$ so that $h(n)=k$ if $d_n\in U_k$. We check that h witnesses that r is below p in the RK-ordering. Since f is an embedding, we have that

$$r=\{Y\subset\omega\colon x=f(u)\in\overline{\bigcup\{U_n\colon n\in Y\}}\}.$$

Hence, for $Y\in r$, the set $h^{-1}[Y]=\{k\colon d_k\in\bigcup\{U_n\colon n\in Y\}\}\in p$. The claim is proved. □

To our knowledge it is an open problem if there may exist such an ultrafilter. So we have to continue.

(iii) Finally to achieve our contradiction we show that if there is a P-point, say r, in ω^*, then u is the limit point of a countable discrete set. Let $x=r\text{-lim}\{d_n\colon n\in\omega\}$ and fix $f\in\text{Emb}$ so that $f(u)=x$. By part (i), we can assume that $f[X]\cap\{d_n\colon n\in\omega\}=\emptyset$. Now, $x\in\overline{\{d_n\colon n\in\omega\}}\cap\overline{(\bigcup_{n\in\omega}f[A_n])}$ and $(\bigcup_{n\in\omega}f[A_n])\cap\{d_n\colon n\in\omega\}=\emptyset$. It follows by Proposition 10, that x is a limit point of the set

$$\bigcup_{n\in\omega}[f[A_n]\cap(\overline{\{d_n\colon n\in\omega\}}-\{d_n\colon n\in\omega\})].$$

But this contradicts that x is a P-point in $\overline{\{d_n\colon n\in\omega\}}-\{d_n\colon n\in\omega\}$. The proof is finished. □

Naturally the previous result raises the following problem which we are unable to solve.

Problem 1. For the systems (X, Emb), with X extremally disconnected, is every universal point isolated?

In this connection, it is interesting to note the following.

Proposition 27 (CH). *Every P-point of ω^* is a universal point in the system* (ω^*, Emb).

Proof. Assume CH and let $p \in \omega^*$ be arbitrary. Rudin [20] showed that the orbit of any P-point of ω^*, under Hom, is exactly the set of all P-points of ω^*. Furthermore, Parovicenko [17] showed that if X is a compact zero-dimensional F-space of weight 2^ω in which each nonempty G_δ has infinite interior, then, under CH, $X \cong \omega^*$. Call such a space X, a Parovicenko space. Therefore it suffices to show that there is a Parovicenko subspace X of ω^*, such that p is a P-point in X. If p is a P-point, there is nothing to do—so assume it is not. Fine and Gillman showed [10] that $\omega^* - \{p\}$ is a zero-dimensional F-space and Gillman [12] showed that there is a partition of $\omega^* - \{p\}$ into relatively clopen sets $U \cup V$ such that $\{p\} = \bar{U} \cap \bar{V}$. Since ω^* is an F-space, p is not the limit point of a σ-compact subset of at least one of U or V, say U. We claim that $X = U \cup \{p\}$ is as desired. Since p is a P-point of X, and U is clopen in $\omega^* - \{p\}$ we can find an increasing chain $\{U_\alpha: \alpha < \omega_1\} \subset \mathrm{CO}(\omega^*)$ such that $U = \bigcup_{\alpha<\omega_1} U_\alpha$. From this it follows easily that X is a Parovicenko space. $\square$

5. Frolík property

In the dynamical system, (X, Emb), we have the notion of the orbit of a point. It is natural to consider the ordering on X induced by these orbits. We may call this the Rudin–Frolík ordering on X.

Definition 28. The Rudin–Frolík pre-ordering $\leqslant$ on a space X is defined by

$$x \leqslant y \quad \text{iff} \quad O(x) \supseteq O(y)$$

in the dynamical system (X, Emb).

Let us recall the classical definition of the Rudin–Frolík ordering. It is defined on the set of ultrafilters on ω, by $p \leqslant q$ if and only if there is a discrete countable set $\{x_n: n \in \omega\}$, such that $q = p\text{-}\lim\{x_n: n \in \omega\}$. As we remarked earlier, the map taking ω to $\{x_n: n \in \omega\}$ uniquely extends to an embedding of $\beta\omega$ into $\beta\omega$ and takes p to q. It seems from the broad applications of the Rudin–Frolík ordering that the embedding is the heart of the notion. This definition only gives a pre-ordering on $\beta\omega$ but it is a partial ordering on the equivalence classes of homeomorphism types.

If $f: \beta\omega \to \beta\omega$ is an embedding, then the set $\{f(n): n \in \omega\}$ is *strongly discrete*, i.e. there are disjoint $A_n \in \mathrm{CO}(\beta\omega)$ such that $f(n) \in A_n$ for each n. From this it is straightforward to verify that if $p \leqslant q$ in the Rudin–Frolík ordering, then $p \leqslant_{\mathrm{RK}} q$. It follows from Rudin's theorem mentioned above that if $p \leqslant q$ and $q \leqslant p$, then $p \approx q$.

For the more general case we will use Frolík's famous theorem about fixed points in compact ED spaces.

Theorem 29 [11]. *Let X be a compact ED space. If $f:X\to X$ is an embedding then*

$$\{x\in X: f(x)=x\} \text{ is clopen.}$$

Corollary 30. *Let $\leqslant$ be the Rudin–Frolík pre-ordering on a compact ED space. If $y\leqslant x$ and $x\leqslant y$, then x and y have the same homeomorphism type, i.e. there is an $h\in \mathrm{Hom}$ such that $h(x)=y$.*

It follows then that the Rudin–Frolík pre-ordering on a compact ED space is a partial ordering on the homeomorphism types. For $x\in X$, we let $[x]$ denote the homeomorphism type of x.

Let us make the following definition.

Definition 31. A space X is said to have the Frolík property if, in the dynamical system (X, Emb), we have for any $x,y\in X$, one of the following conditions holds

$$O(x)\cap O(y)=\emptyset, \qquad O(x)\subseteq O(y), \qquad O(y)\subseteq O(x).$$

Frolík, of course, was the first who proved that $\beta\omega$ has the property defined above. We can make the following observation.

Proposition 32. *X has the Frolík property if and only if, for any $x\in X$,*

$$\{[y]: y\leqslant x\} \text{ is linearly ordered by } \leqslant.$$

Theorem 33. *If X is a compact ccc ED space, then X has the Frolík property.*

This theorem will be deduced as a corollary to the following more basic result. Let us note that a ccc subspace of an ED space is itself ED.

Theorem 34. *Let X be an ED space. If A, B are compact ccc subspaces of X, then $A\cap B$ can be expressed as a union of a clopen subset of A and a clopen subset of B.*

Proof. Note that $A-B$ and $B-A$ are open in $A\cup B$. Let $\{A_n: n\in\omega\}$ be a family of clopen subsets of A whose union is dense in $A-B$ and let $\{B_n: n\in\omega\}$ be a family of clopen subsets of $B-A$ whose union is dense in $B-A$. Since $A\cup B$ is ED, it follows that $\overline{\bigcup_{n\in\omega} A_n}\cap\overline{\bigcup_{n\in\omega} B_n}=\emptyset$. Therefore

$$A\cap B=[A-\overline{\bigcup_{n\in\omega} A_n}]\cup[B-\overline{\bigcup_{n\in\omega} B_n}].$$

Since both A and B are ED, the sets in this union are clopen in A and B respectively. $\square$

Proof of Theorem 33. Let X be a ccc ED space and suppose that $O(x) \cap O(y) \neq \emptyset$. It follows that there are embeddings $f, g \in \text{Emb}$ such that $f(x) = g(y)$. Let $A = f[X]$ and $B = g[X]$. Apply Theorem 34, to obtain $A' \in \text{CO}(A)$ and $B' \in \text{CO}(B)$ such that $A \cap B = A' \cup B'$. Suppose that $f(x) \in A'$. Then $g^{-1} \circ f \restriction f^{-1}[A']$ is an embedding from the clopen set $f^{-1}[A'] \in \text{CO}(X)$ into X which takes x to y. It follows easily, using Corollary 18, that $y \in O(x)$. □

Problem 2. Which spaces have the Frolík property? Is there any compact ED space with uncountable cellularity which has the Frolík property?

We can show, from large cardinal assumptions, that not every compact ED space has the Frolík property. The basic difficulty underlying this discussion is that we have very little information about which compact spaces can be embedded into ED spaces. We do not seem to know anything more than that such spaces must be F-spaces (zero-dimensional). In addition, only one technique for constructing such embeddings has been discovered (independently by Balcar and Simon [4], Kunen, and Shelah). We will introduce it in the proof of the next result.

Theorem 35. *If κ is a measurable cardinal, then $\beta\kappa$ does not have the Frolík property.*

Proof. Let p be a κ-complete free ultrafilter on κ and let $\lambda = 2^\kappa$. We will show that for any $q \in \beta\kappa - \kappa$ such that $\omega \in q$, $O(p) \cap O(q) \neq \emptyset$. However, since p is a P-point of $\beta\kappa$, it is clear that $p \notin O(q)$ and also $q \notin O(p)$ since $\beta\kappa$ obviously cannot be embedded into ω^*.

By Corollary 18, $E(2^\lambda)$ embeds into $\beta\kappa$, hence it will be sufficient to find an embedding of $\beta\kappa$ into $E(2^\lambda)$ so that the image of p is the q-limit of some countable discrete set.

Fix an embedding φ of $\beta\kappa$ into 2^λ such that $z = \varphi(p)$, where z is the constantly zero function. Let $x_\alpha = \varphi(\alpha)$ for each $\alpha \in \kappa$. Recall that 2^λ is a topological group with coordinatewise addition modulo 2. For $A \subset 2^\lambda$ and $x \in 2^\lambda$, $A + x = \{a + x\colon a \in A\}$ is the image of A under the homeomorphism associated with x.

Recall that $E(2^\lambda)$ is the Stone space of $\text{RO}(2^\lambda)$ so we will consider the elements of $E(2^\lambda)$ as ultrafilters on $\text{RO}(2^\lambda)$. Let h denote the canonical mapping from $E(2^\lambda)$ onto 2^λ. Let $\mathcal{U}$ be any member of $h^{-1}(z)$ which is the q-limit of some countable discrete set. For each $\alpha < \kappa$, let

$$\mathcal{U}_\alpha = \{U + x_\alpha\colon U \in \mathcal{U}\}.$$

Since addition by x_α is a homeomorphism, $\mathcal{U}_\alpha \in E(2^\lambda)$ and clearly $h(\mathcal{U}_\alpha) = x_\alpha$.

Since h is continuous and $\overline{\{x_\alpha\colon \alpha \in \kappa\}} \cong \beta\kappa$, which is the maximal compactification of κ, we also have that $\overline{\{\mathcal{U}_\alpha\colon \alpha \in \kappa\}} \cong \beta\kappa$. It remains only to show that $\mathcal{U}$ is the p-limit of $\{\mathcal{U}_\alpha\colon \alpha \in \kappa\}$. That is, let $U \in \mathcal{U}$, we must show that $\{\alpha \in \kappa\colon U \in \mathcal{U}_\alpha\} \in p$. Since 2^λ is ccc and $U \in \text{RO}(2^\lambda)$, there is a countable set $S \subset \lambda$ such that for any

$x,y \in 2^\lambda$, if $x \restriction S = y \restriction S$, then $U+x = U+y$. Now, for each $\xi \in S$, $\{\alpha \in \kappa: x_\alpha(\xi) = z(\xi)\} \in p$. Since p is countably complete, it follows that $\{\alpha \in \kappa: x_\alpha \restriction S = z \restriction S\} \in p$. $\square$

The above embedding result is a special case of [7, Theorem 3.1], see also [8]. Although we cannot prove that, in general, there is a κ such that $\beta\kappa$ does not have the Frolík property, we show that for $\kappa > \omega$, the analogue of the basic Theorem 34, does not hold for $Y_1 = Y_2 = \beta\kappa$.

Example 36. The adjunct space obtained by taking two copies of $\beta\omega_1$ and identifying the two copies of $U(\omega_1)$ via the identity function can be embedded into $\beta\omega_1$.

Proof. We shall work with the space $2^{\omega_1 \cup \mathcal{P}(\omega_1)}$. For each $\alpha \in \omega_1$, we define x^0_α and x^1_α in $2^{\omega_1 \cup \mathcal{P}(\omega_1)}$ as follows.

$$\begin{aligned}
&x^0_\alpha(\beta) = 0 \quad \text{for } \beta \in \omega_1\\
&x^0_\alpha(A) = 1 \quad \text{iff} \quad \alpha \in A \quad \text{for } A \in \mathcal{P}(\omega_1)\\
&x^1_\alpha(\beta) = 0 \quad \text{for } \beta \in \omega_1 - \{\alpha\}\\
&x^1_\alpha(\beta) = 1 \quad \text{for } \beta = \alpha\\
&x^1_\alpha(A) = 1 \quad \text{iff} \quad \alpha \in A \quad \text{for } A \in \mathcal{P}(\omega_1)
\end{aligned}$$

Choose any ultrafilter $\mathcal{U}$ on $\mathrm{RO}(2^{\omega_1 \cup \mathcal{P}(\omega_1)})$ which converges to the constant zero function, i.e. whose image under the canonical mapping is the constantly zero function. For each $\alpha < \omega_1$ and $i = 0,1$, let

$$\mathcal{U}^i_\alpha = \{U + x^i_\alpha : U \in \mathcal{U}\}.$$

As above, it can be shown that, for each $i = 0,1$, the closure in $E(2^{\omega_1 \cup \mathcal{P}(\omega_1)})$, of $\{\mathcal{U}^i_\alpha : \alpha \in \omega_1\}$ is a copy of $\beta\omega_1$. Furthermore, it is clear that for each $\alpha < \omega_1$, $\overline{\{\mathcal{U}^0_\gamma : \gamma < \alpha\}}$ is disjoint from $\overline{\{\mathcal{U}^1_\gamma : \gamma < \alpha\}}$. Let $p \in \beta\omega_1$ be any uniform ultrafilter, i.e. for each $\alpha \in \omega_1$, the set $\omega_1 - \alpha$ is a member of p. Let $\mathcal{W}$ be the p-limit of $\{\mathcal{U}^0_\alpha : \alpha \in \omega_1\}$. We finish by showing that $\mathcal{W}$ is also the p-limit of $\{\mathcal{U}^1_\alpha : \alpha < \omega_1\}$. Let $W \in \mathcal{W}$ be arbitrary. Let y be the element of $2^{\omega_1 \cup \mathcal{P}(\omega_1)}$ to which $\mathcal{W}$ converges. Let a countable set $S \subset \omega_1 \cup \mathcal{P}(\omega_1)$ be chosen as above, i.e. if $x \restriction S = x' \restriction S$, then $W + x = W + x'$. Since $\mathcal{W}$ is the p-limit of $\{\mathcal{U}^0_\alpha : \alpha \in \omega_1\}$, there is a set $A \in p$ such that $W \in \mathcal{U}^0_\alpha$ for each $\alpha \in A$. We will show that $W \in \mathcal{U}^1_\alpha$ for each $\alpha \in A - S$. Since p is a uniform ultrafilter, this will complete the proof. Let $\alpha \in A - S$ and observe that $W + x^0_\alpha = W + x^1_\alpha$. Since $W \in \mathcal{U}^0_\alpha$, we have that $W + x^0_\alpha \in \mathcal{U}$, hence $W + x^1_\alpha \in \mathcal{U}$. Now observe that $W = (W + x^1_\alpha) + x^1_\alpha \in \mathcal{U}^1_\alpha$. This completes the proof. $\square$

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